

A Note on Altermatic Number

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Abstract

In view of Tucker's lemma (an equivalent combinatorial version of the Borsuk-Ulam theorem), the present authors (2013) introduced the k^{th} altermatic number of a graph G as a tight lower bound for the chromatic number of G . In this note, we present a purely combinatorial proof for this result.

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1 Introduction

Throughout the paper, the set $\{1, 2, \dots, n\}$ is denoted by $[n]$. A *hypergraph* \mathcal{H} is an ordered pair $(V(\mathcal{H}), E(\mathcal{H}))$, where $V(\mathcal{H})$ and $E(\mathcal{H})$ are the vertex set and the hyperedge set of \mathcal{H} , respectively. Hereafter, all hypergraphs are simple, i.e., $E(\mathcal{H})$ is a family of distinct nonempty subsets of $V(\mathcal{H})$. A proper t -coloring of a hypergraph \mathcal{H} , is a mapping $c : V(\mathcal{H}) \rightarrow [t]$ such that no hyperedge is monochromatic, i.e., $|c(e)| > 1$ for any hyperedge $e \in E(\mathcal{H})$. The minimum integer t such that \mathcal{H} admits a t -coloring is called the *chromatic number* of \mathcal{H} and is denoted by $\chi(\mathcal{H})$. For a hypergraph containing some hyperedge of cardinality 1, we define its chromatic number to be infinite.

For a vector $X = (x_1, x_2, \dots, x_n) \in \{R, 0, B\}^n$, a subsequence $x_{a_1}, x_{a_2}, \dots, x_{a_t}$ ($1 \leq a_1 < a_2 < \dots < a_t \leq n$) of nonzero terms of X is called an *alternating subsequence* of X if any two consecutive terms in this subsequence are different. We denote by $alt(X)$ the length of a longest alternating subsequence of X . Moreover, we define $alt(0, 0, \dots, 0) = 0$. Also, we denote the number of nonzero terms of X by $|X|$. For instance, if $X = (R, R, B, B, 0, R, 0, R, B)$, then $alt(X) = 4$ and $|X| = 7$. For an $X = (x_1, x_2, \dots, x_n) \in \{R, 0, B\}^n$, define $X^R = \{i : x_i = R\}$ and $X^B = \{i : x_i = B\}$. Note that if we consider a vector $X = (x_1, x_2, \dots, x_n)$, then one can obtain X^R and X^B , and conversely. Therefore, by abuse of notation, we can set $X = (X^R, X^B)$. Throughout the paper, we use interchangeably these representations, i.e., $X = (x_1, x_2, \dots, x_n)$ or $X = (X^R, X^B)$. For $X = (X^R, X^B), Y = (Y^R, Y^B) \in$

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$\{R, 0, B\}^n$, we write $X \subseteq Y$, if $X^R \subseteq Y^R$ and $X^B \subseteq Y^B$. Note that if $X \subseteq Y$, then every alternating subsequence of X is an alternating subsequence of Y , and subsequently, $\text{alt}(X) \leq \text{alt}(Y)$. Also, if the first nonzero term of X is R (resp. B), then every alternating subsequence of X with the maximum length begins with R (resp. B), and moreover, we can conclude that X^R (resp. X^B) contains the smallest integer of $X^R \cup X^B$.

Let $L_{V(\mathcal{H})} = \{v_{i_1} < v_{i_2} \dots < v_{i_n} : (i_1, i_2, \dots, i_n) \in S_n\}$ be the set of all linear orderings of the vertex set of hypergraph \mathcal{H} , where $V(\mathcal{H}) = \{v_1, v_2, \dots, v_n\}$. For any $X = (x_1, x_2, \dots, x_n) \in \{R, 0, B\}^n$ and any linear ordering $\sigma : v_{i_1} < v_{i_2} \dots < v_{i_n} \in L_{V(\mathcal{H})}$, define $X_\sigma^R = \{v_{i_j} : x_j = R\}$, $X_\sigma^B = \{v_{i_k} : x_k = B\}$, and $X_\sigma = (X_\sigma^R, X_\sigma^B)$. Note that for $V = [n]$ and $I : 1 < 2 < \dots < n$, we have $X^R = X_I^R$, $X^B = X_I^B$, and $X = X_I$. Also, set $\mathcal{H}_{|X_\sigma}$ to be the hypergraph with the vertex set $X_\sigma^R \cup X_\sigma^B$ and the edge set

$$E(\mathcal{H}_{|X_\sigma}) = \{A \in E(\mathcal{H}) : A \subseteq X_\sigma^R \text{ or } A \subseteq X_\sigma^B\}.$$

For a hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$, the *general Kneser graph* $\text{KG}(\mathcal{H})$ has all hyperedges of \mathcal{H} as vertex set and two vertices of $\text{KG}(\mathcal{H})$ are adjacent if the corresponding hyperedges are disjoint. The hypergraph \mathcal{H} provides a *Kneser representation* for a graph G whenever G and $\text{KG}(\mathcal{H})$ are isomorphic. It is simple to see that a graph G has various Kneser representations. For any $\sigma \in L_{V(\mathcal{H})}$ and positive integer k , define $\text{alt}_\sigma(\mathcal{H}, k)$ to be the largest integer t such that there exists an $X \in \{R, 0, B\}^n$ with $\text{alt}(X) = t$ and that the chromatic number of $\text{KG}(\mathcal{H}_{|X_\sigma})$ is at most $k - 1$ (For $k = 1$, it means that $\mathcal{H}_{|X_\sigma}$ contains no hyperedge). Now define

$$\text{alt}(\mathcal{H}, k) = \min \{\text{alt}_\sigma(\mathcal{H}, k) : \sigma \in L_{V(\mathcal{H})}\}.$$

Let G be a graph and k be a positive integer such that $1 \leq k \leq \chi(G) + 1$. The k^{th} *altermatic number* of G , $\zeta(G, k)$, is defined as follows

$$\zeta(G, k) = \max_{\mathcal{H}} \{|V(\mathcal{H})| - \text{alt}(\mathcal{H}, k) + k - 1 : \text{KG}(\mathcal{H}) \longleftrightarrow G\}$$

where $\text{KG}(\mathcal{H}) \longleftrightarrow G$ means there are some homomorphisms from G to H and also from H to G .

One can see that for $k = \chi(\text{KG}(\mathcal{H})) + 1$, we have $\text{alt}(\mathcal{H}, k) = |V|$; and consequently, $\chi(\text{KG}(\mathcal{H})) = \zeta(G, k)$. In [1], in view of Tucker's lemma, it was shown that the k^{th} altermatic number of a graph is a tight lower bound for its chromatic number.

Theorem A. [1] *For any graph G and positive integer k , where $k \leq \chi(G) + 1$, we have $\chi(G) \geq \zeta(G, k)$.*

Furthermore, it was shown [1] that the first altermatic number can be considered as an improvement of the Dol'nikov-Kříž's lower bound [3, 5] for the chromatic number of general Kneser hypergraphs. Also, we should mention that by an improvement of Gale's lemma, in [2], it was shown that the first altermatic number provides tight lower bound for some well-known topological parameter which is related to graphs via the Borsuk-Ulam theorem.

For more details about Tucker's lemma and Gale's lemma, we refer readers to [2, 7]. In [9], Meunier showed that it is a hard problem to determine the exact value of

the first altermatic number of a graph. Precisely, he proved that for a hypergraph \mathcal{H} and a permutation σ of $V(\mathcal{H})$, it is an NP-hard problem to specify $alt_\sigma(\mathcal{H}, 1)$.

2 A Combinatorial Proof of Theorem A

For two positive integers m and n where $m \geq 2n$, the usual Kneser graph $KG(m, n)$ is a graph whose vertex set consists of all n -subsets of $[m]$ and two vertices are adjacent if the corresponding subsets are disjoint. Kneser in 1955 [4] conjectured that $\chi(KG(m, n)) \geq m - 2n + 2$. In 1978 [6], Lovász by using algebraic topology, proved Kneser conjecture. Next Schrijver [10] introduced the Schrijver graph $SG(m, n)$ as a subgraph of $KG(m, n)$ and proved that it is critical and has the same chromatic number as $KG(m, n)$. It is proved in [1] that Theorem A (for $k = 1, 2$) implies these results. In what follows, we present the proof of Theorem A. It should be mentioned that the proof relies on an interesting idea used by Matoušek [8] to present a combinatorial proof of Lovász-Kneser theorem.

Proof of Theorem A. On the contrary, suppose $\zeta(G, k) > \chi(G)$. Consider a hypergraph \mathcal{H} such that $KG(\mathcal{H})$ is isomorphic to G and that

$$\zeta(G, k) \geq |V(\mathcal{H})| - alt(\mathcal{H}, k) + k - 1 = |V(\mathcal{H})| - alt_\sigma(\mathcal{H}, k) + k - 1 > \chi(G)$$

where $\sigma \in L_{V(\mathcal{H})}$. Without loss of generality and for the simplicity of notations, we may assume that $V = [n]$ and $\sigma = I : 1 < 2 < \dots < n$. Let $h : V(KG(\mathcal{H})) = E(\mathcal{H}) \rightarrow \{1, 2, \dots, n - alt(\mathcal{H}, k) + k - 2\}$ be a proper coloring of $KG(\mathcal{H})$. For any subset $M \subseteq V(\mathcal{H})$, we define $\bar{h}(M) = \max\{h(A) : A \subseteq M, A \in E(\mathcal{H})\}$. If there is no $A \subseteq M$, where $A \in E(\mathcal{H})$, then set $\bar{h}(M) = 0$. For $X = (X^R, X^B) \in \{R, 0, B\}^n$, set $\bar{h}(X) = \max\{\bar{h}(X^R), \bar{h}(X^B)\}$. Define a map $\lambda : \{R, 0, B\}^n \rightarrow \{\pm 1, \pm 2, \dots, \pm n\}$ as follows

- If $X = (X^R, X^B) \in \{R, 0, B\}^n$ and $alt(X) \leq alt_I(\mathcal{H}, k)$, set

$$\lambda(X) = \begin{cases} +alt(X) + 1 & \text{if } X^B = \emptyset \text{ or } \min(X^R \cup X^B) \in X^R \\ -alt(X) - 1 & \text{otherwise} \end{cases}$$

- If $X = (X^R, X^B) \in \{R, 0, B\}^n$ and $alt(X) \geq alt_I(\mathcal{H}, k) + 1$, set

$$\lambda(X) = \begin{cases} alt_I(\mathcal{H}, k) + \bar{h}(X) - k + 2 & \text{if } \bar{h}(X) = \bar{h}(X^R) \\ -(alt_I(\mathcal{H}, k) + \bar{h}(X) - k + 2) & \text{if } \bar{h}(X) = \bar{h}(X^B) \end{cases}$$

Since h is a proper coloring, one can see that the map λ is well-defined. Moreover, it is straightforward to check that for any two ordered pairs $(A_i, B_i) \subseteq (A_j, B_j)$, we have $\lambda((A_i, B_i)) + \lambda((A_j, B_j)) \neq 0$. Also, in view of the definition of $alt_I(\mathcal{H}, k)$, if $alt(X) \geq alt_I(\mathcal{H}, k) + 1$, then the chromatic number of $KG(F|_X)$ is at least k , and consequently, $|\lambda(X)| \geq alt_I(\mathcal{H}, k) + 2$. Also, by the definition of λ , one can see that $\lambda((\emptyset, \emptyset)) = 1$. In the sequel, we show that there exists a graph H with a unique vertex of degree one and any other vertex of degree 2, which is impossible. This contradicts our assumption that h is a proper coloring. For any subset $A \subseteq [n]$, define $-A = \{-t : t \in A\}$. Note that if $A = \emptyset$, then $-A = \emptyset$. A permissible

sequence is a sequence $(A_0, B_0) \subseteq (A_1, B_1) \subseteq \dots \subseteq (A_m, B_m)$ of disjoint ordered pairs of $\{R, 0, B\}^n$ such that for any $0 \leq i \leq m$, we have $|A_i| + |B_i| = i$ and that $A_m \cup -B_m \subseteq \{\lambda((A_0, B_0)), \lambda((A_1, B_1)), \dots, \lambda((A_m, B_m))\}$. By definition, we have $A_0, B_0 = \emptyset$ and $0 \leq m \leq n$. Also, one can see that (\emptyset, \emptyset) is a permissible sequence.

Set the vertex set of the graph H to be the set of all permissible sequences. Now we introduce the edge set of H . For the permissible sequence (\emptyset, \emptyset) , we define its unique neighbor to be $(\emptyset, \emptyset) \subseteq (\{1\}, \emptyset)$, which is a permissible sequence. We assign two neighbors to any other permissible sequence as follows. Moreover, we show that this assignment is symmetric, i.e., H is an undirected graph. Consider a permissible sequence $(A_0, B_0) \subseteq (A_1, B_1) \subseteq \dots \subseteq (A_m, B_m)$ ($m \geq 1$) and set $\lambda_i = \lambda((A_i, B_i))$ for any $0 \leq i \leq m$. By the definition of λ , it is clear that if $(A, B) \subseteq (A', B')$, then $|\lambda(A, B)| \leq |\lambda(A', B')|$. Therefore, in view of the definition of permissible sequence, one of the following conditions holds

- (i) There exists a unique integer $0 \leq i < m$ such that $\lambda_i = \lambda_{i+1}$.
- (ii) There exists a unique integer $0 \leq i \leq m$ such that $\lambda_i \notin A_m \cup -B_m$.

Note that in case (i), $i = 0$ is not possible since for any $j > 0$, $|\lambda_j| > 1$. If $\lambda_i = \lambda_{i+1}$ for $1 \leq i < m$, then we define two neighbors of $(A_0, B_0) \subseteq (A_1, B_1) \subseteq \dots \subseteq (A_m, B_m)$ as follows.

1. $(A'_0, B'_0) \subseteq (A'_1, B'_1) \subseteq \dots \subseteq (A'_m, B'_m)$, where $(A'_i, B'_i) = (A_{i-1} \cup (A_{i+1} \setminus A_i), B_{i-1} \cup (B_{i+1} \setminus B_i))$ and for any $r \neq i$, $(A'_r, B'_r) = (A_r, B_r)$.
2. If $i < m - 1$, then define the other neighbor to be $(A''_0, B''_0) \subseteq (A''_1, B''_1) \subseteq \dots \subseteq (A''_m, B''_m)$, where $(A''_{i+1}, B''_{i+1}) = (A_i \cup (A_{i+2} \setminus A_{i+1}), B_i \cup (B_{i+2} \setminus B_{i+1}))$ and for any $r \neq i + 1$, $(A''_r, B''_r) = (A_r, B_r)$. Otherwise, if $i = m - 1$, define the other neighbor to be $(A_0, B_0) \subseteq (A_1, B_1) \subseteq \dots \subseteq (A_{m-1}, B_{m-1})$.

One can check that both of neighbors are permissible. Now suppose that there exists an integer $0 \leq i \leq m$ such that $\lambda_i \notin A_m \cup -B_m$. Define the neighbors as follows

1. $(A'_0, B'_0) \subseteq (A'_1, B'_1) \subseteq \dots \subseteq (A'_m, B'_m) \subseteq (A'_{m+1}, B'_{m+1})$, where for any $0 \leq r \leq m$, $(A'_r, B'_r) = (A_r, B_r)$ and if $\lambda_i > 0$, then $(A'_{m+1}, B'_{m+1}) = (A_m \cup \{\lambda_i\}, B_m)$, otherwise, $(A'_{m+1}, B'_{m+1}) = (A_m, B_m \cup \{-\lambda_i\})$.
2. If $1 \leq i \leq m - 1$, then define the other neighbor to be $(A''_0, B''_0) \subseteq (A''_1, B''_1) \subseteq \dots \subseteq (A''_m, B''_m)$, where $(A''_i, B''_i) = (A_{i-1} \cup (A_{i+1} \setminus A_i), B_{i-1} \cup (B_{i+1} \setminus B_i))$ and for any $r \neq i$, $(A''_r, B''_r) = (A_r, B_r)$. If $i = m$, define the second neighbor to be $(A_0, B_0) \subseteq (A_1, B_1) \subseteq \dots \subseteq (A_{m-1}, B_{m-1})$. Otherwise, for $i = 0$, consider $(B_0, A_0) \subseteq (B_1, A_1) \subseteq \dots \subseteq (B_m, A_m)$ as the second neighbor.

Note that all neighbors are permissible. Also, one can check that the aforementioned assignment is symmetric, which completes the proof. ■

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